# Sum of Binary Trailing Zeros

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*[2018-05-15 Tue 20:46]*

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While working through a project euler problem I noticed an interesting pattern:

When writing down the numbers from 1 to $n$ the total number of trailing zeros (called $S(n)$ for the rest of this post) is equal to $n - popcount(n)$ where $popcount(n)$ is the number of ones in the binary representation of $n$.

It took me a while to understand why this is true and even longer to formulate a somewhat elegant proof. What follows is my best try so far.

## 1 Example

1. 0001
2. 001 **0**
3. 0011
4. 01 **00**
5. 0101
6. 011 **0**
7. 0111
8. 1 **000**

9. 1001

10. 101 **0**

In the first $n = 10$ numbers there are 8 trailing zeros. $popcount(n) = 2$, so the equation holds.

## 2 Proof

A binary number has at least $z$ trailing zeros $\iff$ it is a multiple of $2^z$.

In the example above there are $\left\lfloor \frac{10}{2^1} \right\rfloor = 5$ numbers with at least one trailing zero, $\left\lfloor \frac{10}{2^2} \right\rfloor = 2$ numbers with at least two trailing zeros, and $\left\lfloor \frac{10}{2^3} \right\rfloor = 1$ numbers with at least three trailing zeros.

For arbitrary $n$ this can be written as

$$S(n) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor$$

which is not really an infinite sum because after a while ($i > \log_2(n)$) all the remaining summands are zero.

$n$ can be written (binary notation) as $\sum\limits_{j=0}^{\infty} a_j 2^j$ with $a_j \in \{0, 1\}$.

$$S(n) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{2^i} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \frac{\sum\limits_{j=0}^{\infty} a_j 2^j}{2^i} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \sum_{j=0}^{\infty} a_j 2^{j-i} \right\rfloor$$

The division by $2^i$ can be understood as moving the "decimal" point in the binary representation of $n$ $i$ places to the left. $\lfloor x \rfloor$ is equivalent to cutting off everything after the "decimal" point of $x$ (independent of the base).

$$\begin{aligned} S(10) = &\sum_{i=1}^{\infty} \left\lfloor \frac{1010_2}{2^i} \right\rfloor \\ &= \lfloor 101.0_2 \rfloor + \lfloor 10.10_2 \rfloor + \lfloor 1.010_2 \rfloor + \lfloor 0.1010_2 \rfloor + \lfloor 0.01010_2 \rfloor + \ldots + \lfloor 0.0 \ldots 01010_2 \rfloor \\ &= 101_2 + 10_2 + 1_2 + 0_2 + 0_2 + \ldots + 0_2 \\ &= 1000_2 = 8 \end{aligned}$$

How does this apply to the equation from before? First I'll introduce a simple indicator function that will make the next steps of the proof a little bit easier to follow:

$$I(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

What this allows us to do is to rewrite $\lfloor \dots \rfloor$ by multiplying each $a_i 2^{j-i}$ with $I(j-i)$ so that it removed from the sum if $2^{j-i} < 1$ (as part of the numbers behind the "decimal" point that are cut of).

$$\begin{aligned}
S(n) &= \sum_{i=1}^{\infty} \left\lfloor \sum_{j=0}^{\infty} a_j 2^{j-i} \right\rfloor \\
&= \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} a_j 2^{j-i} I(j-i) \\
&= \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} a_j 2^{j-i} I(j-i) && \text{(swapping the sums)} \\
&= \sum_{j=0}^{\infty} a_j \sum_{i=1}^{\infty} 2^{j-i} I(j-i) \\
&= \sum_{j=0}^{\infty} a_j \sum_{i=1}^{j} 2^{j-i} && \text{(by definition of } I(x)\text{)} \\
&= \sum_{j=0}^{\infty} a_j \sum_{i=0}^{j-1} 2^{j} \\
&= \sum_{j=0}^{\infty} a_j (2^j - 1) \\
&= \sum_{j=0}^{\infty} a_j 2^j - \sum_{j=0}^{\infty} a_j \\
&= n - \mathit{popcount}(n)
\end{aligned}$$

If the step from $\sum_{i=0}^{j-1} 2^i$ to $2^j - 1$ is not obvious, think about what happens when 1 is added to a binary number that is made up entirely of ones: $1111_2 + 1_2 = 10000_2$.

## 3 Trailing Zeros of $n!$

Let $T(n)$ be the number of trailing zeros when $n$ is written in binary.

$T(ab) = T(a) + T(B)$ (see Binary Trailing Zeros of Products), so $T(n!) = \sum_{i=1}^{n} T(n) = S(n)$.

Example: $10! = 3628800 = 1101110101111100000000_2$ has $S(10) = 10 - 2 = 8$ trailing zeros.